Homework 3
Math 117 - Summer 2022

1) (4 points) Let V be finite dimensional and let $W \subseteq V$ be a subspace. Recall the definition of the annihilator of W, W^o from class. Prove using dual basis that

$$dim(W^o) = dim(V) - dim(W)$$

(hint: extend basis....)

Solution:

2) (3 points) Let V be any vector space (potentially infinite dimensional). Prove that

$$(V/W)^* \simeq W^o$$

(Hint: Universal property of quotient....)
Remark: This isomorphism gives another proof of problem 1, in the case when V is finite dimensional

Solution:

3) (3 points) Let V, W be finite dimensional vector spaces over $\mathbb{F}$ and let $T : V \to W$ be a linear map. Recall the isomorphism we constructed in class $\Phi_V : V \to V^{**}$ by sending v to ev_v. Prove that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\Phi_V} & V^{**} \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T^{**}} \\ W & \xrightarrow{\Phi_W} & W^{**} \end{array}$$

ie, that $\Phi_W \circ T = T^{**} \circ \Phi_V$ (Hint: Recall that $T^{**} : V^{**} \to W^{**}$ sends a linear functional $\varphi : V^* \to \mathbb{F}$ to the linear functional $\varphi \circ T^* : W^* \to \mathbb{F}$. That is $T^{**}(\varphi) = \varphi \circ T^* \in W^{**}$. You will then evaluate what this is on a linear functional $\gamma \in W^*$)

Solution:

4) Let V be an n-dimensional vector space. We call a subspace of dimension n-1 a hyperplane.

(a) (1 point) If $\varphi : V \to \mathbb{F}$ is a nonzero linear functional, prove that $ker(\varphi)$ is a hyperplane

(b) (2 points) Prove moreover that every hyperplane is the kernal of a nonzero linear functional.

(c) (2 points) More generally, prove that a subspace of dimension d is the intersection of n-d hyperplanes (ie, from part b, is the intersection of n-d kernals of linear functionals). (Hint: Dual basis can be helpful here...)

Solution:

5) Let V, W be finite dimensional vector spaces over $\mathbb{F}$.

(a) (3 points) Prove that

$$V \otimes W^* \simeq \mathcal{L}(V, W)$$

(b) (2 points) Use this to prove that

$$(V \otimes W)^* \simeq \mathcal{L}(V, W^*)$$

Unimportant Remark: Writing out the duals, this isomorphism above is saying that

$$\mathcal{L}(V \otimes W, \mathbb{F}) \simeq \mathcal{L}(V, \mathcal{L}(W, \mathbb{F}))$$

That is, maps out of the tensor product of V and W into $\mathbb{F}$ correspond to maps from V into maps from W to $\mathbb{F}$. Such a result is in fact true more generally if we replace $\mathbb{F}$ with any other vector space, and is a foundational result in category theory/algebra: the so called "tensor-hom adjunction." Cool stuff

Solution:

5') (You can either do the 5 above or this problem)
Prove that f and g are inverses of each for the examples we did in class:

(a)

$$f : V \otimes W \to W \otimes V \text{ that sends the simple tensor}$$
$$v \otimes w \to w \otimes v$$
$$g : W \otimes V \to V \otimes W \text{ that sends the simple tensor}$$
$$w \otimes v \to v \otimes w$$

(b)

$$f : (V_1 \oplus V_2) \otimes W \to (V_1 \otimes W) \oplus (V_2 \otimes W)$$
$$g : (V_1 \otimes W) \oplus (V_2 \otimes W) \to (V_1 \oplus V_2) \otimes W$$

as we defined in the notes

(c)

$$f : (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3)$$
$$g : V_1 \otimes (V_2 \otimes V_3) \to (V_1 \otimes V_2) \otimes V_3$$

where in this case define the inverse map g as hinted in the notes and then prove it actually is the inverse

6) Consider the following vector spaces with corresponding basis:

$$V_1 = \mathbb{R}^3 \qquad \mathcal{B}_{V_1} = \{e_1, e_2, e_3\}$$
$$W_1 = \mathbb{R}[t]_{\leq 2} \qquad \mathcal{B}_{W_1} = \{1, t, t^2\}$$
$$V_2 = M_{2\times 2}(\mathbb{R}) \qquad \mathcal{B}_{V_2} = \{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\}$$
$$W_2 = M_{2\times 2}(\mathbb{R}) \qquad \mathcal{B}_{W_2} = \{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\}$$

Now consider the following two linear transformations $T_1 : V_1 \to W_1$ and $T_2 : V_2 \to W_2$ given by

$$T_1(\begin{pmatrix}a\\b\\c\end{pmatrix}) = a + b - ct + at^2$$
$$T_2\left(\begin{pmatrix}a_1 & a_2\\a_3 & a_4\end{pmatrix}\right) = \begin{pmatrix}2a_2 & a_4\\a_1 & 3a_3\end{pmatrix}$$

(a) (1 point) Write the corresponding basis for $V_1 \otimes V_2$ and $W_1 \otimes W_2$

(b) (4 points) Recall we get the linear map

$$T_1 \otimes T_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$$
$$(T_1 \otimes T_2)(v_1 \otimes v_2) = T_1(v_1) \otimes T_2(v_2)$$

Compute the matrix of this map with respect to the two basis you found in part a

Solution:

Unimportant Rmk: This is an example of what is called the *Kronecker-Product* of matrices. It is an operation that takes an $m \times n$ and a $k \times l$ matrix and produces an $mk \times nl$ matrix. This matrix is precisely the matrix of the tensor product of linear maps we defined in class/on your mini-hw. I recommend looking it up- its a pretty cool thing.